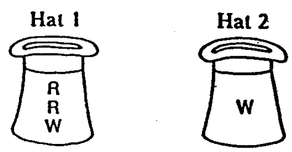
AREA MODELS

In this activity, students are introduced to probabilistic situations in which one event depends on another event.

There are two major parts. The first part involves a maze that starts at a common point but can end in one of two rooms. If paths are chosen randomly, students must decide which room is most likely to be the room in which the maze ends. Ending up in one of the rooms depends upon choosing a succession of paths.

The second part involves selecting a hat and then picking a marble from the hat. The challenge is to arrange two black and two white marbles in the two hats to maximize the probability of drawing a white marble. Drawing a marble out of a hat depends upon first choosing a hat.

In these examples a simple listing of the possible events does not help very much because the outcomes are not equally likely. For instance, in the four marbles in two hats example the students have to find the probability of drawing a white marble in one draw from one of the two hats chosen at random. Suppose the marbles are placed as shown below.

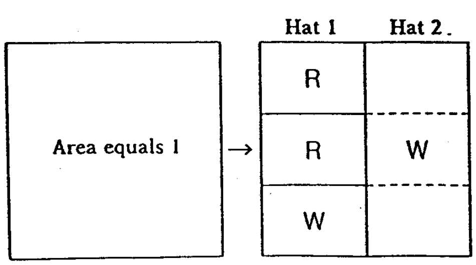


If we tried to calculate P(W) directly, we would have to multiply probabilities and then add probabilities

P(H1 and W) + P(H2 and W)

$$\left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{2} \times 1\right) = \frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$$

This approach is very difficult for most students to understand. That it must be made for all the possible arrangements of the marbles makes this analysis even more confusing. However, using an area model to analyze the situation makes the problem quite accessible to students in grades 6, 7, and 8.



If the total area is 1, then W represents $\frac{4}{6}$ or $\frac{2}{3}$ of the area. Hence P(W) = $\frac{2}{3}$.

This type of analysis is very useful in situations requiring more than one choice to complete an event and the choices available depend on what happened before.

Activity 6

Goals for students

1. Organize data.
2. Understand the phrase "to choose at random."
3. Partition the area of a square so that the parts will represent the probabilities of every possible outcome.
4. Using an area model, analyze a situation to determine theoretical probability.
5. For appropriate grade levels, relate the area model to multiplying probabilities.

Materials

Coin for choosing paths on a maze (optional).
Two colored pencils (for teacher transparency).
Two colored pencils for each student (optional).
*Materials 6-1, The Maze.
*Materials 6-2, Another Maze.

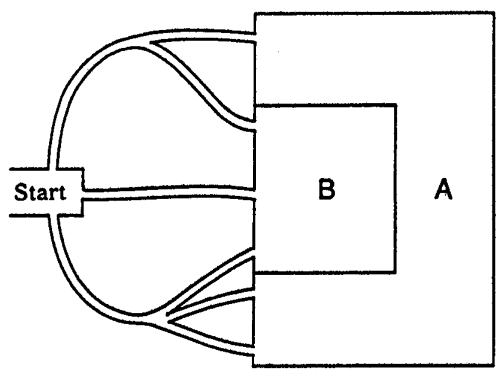
Worksheets

*6-1, Which Is Best?
*6-2, Darts Anyone?

Transparencies

Starred items should be made into transparencies.

TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE
Tell a story.	Mr. Icky Green is the dietitian at a middle school down the road. His favorite dish is Cheese and Rutabaga Surprise, which he serves twice a week in the school cafeteria. One day a group of students who were fed up with C and R Surprise approached Mr. Green with this proposition: The students would place a new cookbook in one of two rooms at the end of a maze. Mr. Green would walk the maze, randomly choosing a path at each intersection. If he enters the room with the cookbook, no more Cheese and Rutabaga Surprise. If he enters the empty room, C and R Surprise will appear with disgusting regularity for the rest of the year.	
	In which room should the students place the cookbook to have the best chance of being found by Mr. Green?	Various answers and reasons.
Display a transparency of the Maze (Materials 6-1).	This is a map of the maze.	



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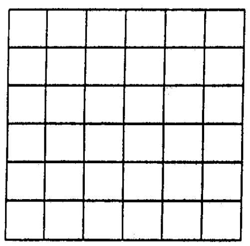
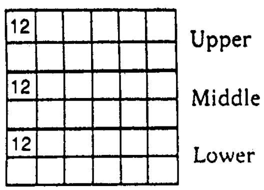
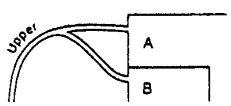
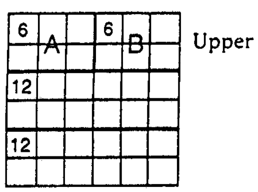
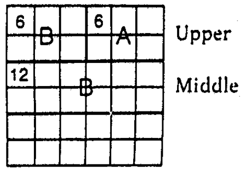
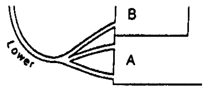
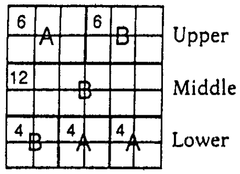
TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE
Let several students give an answer and reasons for that answer. Here you may want to ask the students to devise a way for Mr. Green to make his choices so that they will be *chosen at random*. For example, toss a die. For a three-way decision: left 1, 2; middle 3, 4; right 5, 6. For a two-way decision: left 1, 2, 3; right 4, 5, 6. After the students have discussed their choices, suggest that a careful analysis to find the probability that Mr. Green ends up in room A and in room B is needed.	Why do you think the students should choose the room you have suggested?	If the students raise such issues as "I always choose the middle one if there are three choices," remind them that Mr. Green will choose *at random*. He is equally likely to choose any one of the three. Some students may suggest that B is the proper choice, because you can reach B from each of the three paths, but A can be reached from only two. Others may say B, because the middle path leads directly to it. Some will say they are equal, because there are three doors for each room.
Use the 6 × 6 grid on the transparency of Materials 6-1. Illustrate. The area of the grid is $\frac{36}{36}$. 	We can use this square to help us analyze the students' problem. To make partitioning easier we'll start with a square marked off into 36 smaller squares. The area of the large square, which is 1, is represented as $\frac{36}{36}$. We will trace each path Mr. Green could take and represent the results with an area that represents the probability of the path being taken.	

TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE
	The first choice he has to make is between the upper, middle, and lower paths.	
	How should we partition the area of the square to represent the probabilities that should be assigned for this first choice Mr. Green must make?	Partition the area into three equal parts.
Make partitions with two colored pens. Write down the number of squares in each region as you partition.	Let's partition the area with red horizontal lines and label the paths. How many small squares should be in each part?	12
	Now let's analyze each path that we have started. First, the upper path splits into two paths, one leading to A and one to B.	
	How should we partition the area of the upper path to show this split?	Partition it into two equal areas.
	We will use a green vertical line through the area for the upper path to show the split. Let's label each part with the letter of the room to which the path leads.	
	How many squares are in each of the new areas?	6
	In the same way let's analyze the middle and lower paths. Should the middle path be partitioned?	No; it leads only to room B.

TEACHER ACTION | **TEACHER TALK** | **EXPECTED RESPONSE**



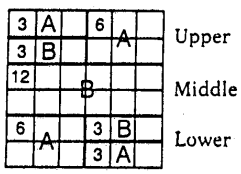


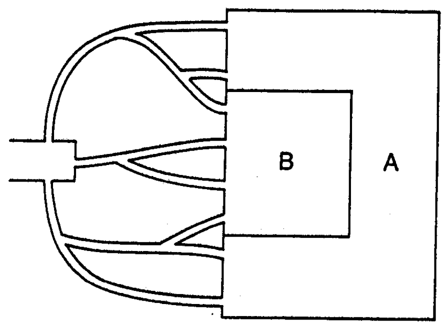


TEACHER TALK	EXPECTED RESPONSE
Then let's label the middle area B.	
How should we partition the area for the lower path?	We should partition the area into 3 equal parts and mark one B and two A.
How many small squares are in each part?	4
In all, how many small squares represent ending in room A? How many represent ending in room B?	14 squares for A; 22 squares for B.
How would we express this as a probability?	$P(A) = \frac{14}{36} = \frac{7}{18}$ $P(B) = \frac{22}{36} = \frac{11}{18}$
Which room are you most likely to enter?	Room B.

Display a transparency of Another Maze (Materials 6-2). Have students vote on which room is most likely to be entered.

Repeat analysis of Mr. Green and the students with a new diagram of the maze.





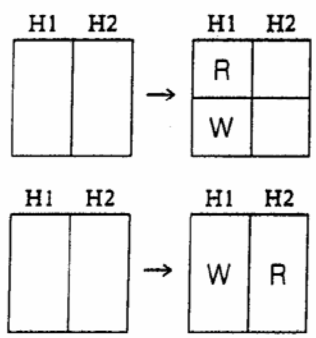
TEACHER TALK	EXPECTED RESPONSE
What is the probability that Mr. Green will enter room A? What is the probability that Mr. Green will enter room B?	$P(A) = \frac{18}{36} = \frac{1}{2}$ $P(B) = \frac{18}{36} = \frac{1}{2}$
In which room should the students place the cookbook?	Either.

TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE
Tell the story.	Here is another problem to consider.	
	We are given an opportunity to win a prize. We are given two hats and two marbles, one red marble and one white marble. We may arrange the marbles in the hats any way we choose. Then another person will randomly draw one marble out of one of the hats. If the marble is white we'll win the prize. How should we arrange the marbles?	Various answers.
Illustrate.	Let's list all the possibilities. There are essentially only two.	
	1. Both marbles are in one of the hats.	
	2. One marble is in each hat.	
	We will use an area model to find P(W) in each case.	
Illustrate.	First we choose a hat H1 or H2. Each has an equal chance. Then we partition the area for each hat to represent the probability of drawing a red or a white marble from that hat.	
	What is the probability of drawing either the red or white marble in case 1?	$P(R) = \frac{1}{4}$ $P(W) = \frac{1}{4}$
	What is the probability of drawing either the red or white marble in case 2?	$P(R) = \frac{1}{2}$ $P(W) = \frac{1}{2}$
	Which arrangement is the best?	Case 2.

Case 1 Case 2

Case 1

Case 2



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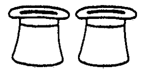
TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE
Tell a story. Illustrate.	Now, suppose that you and a friend are given an opportunity to win a prize. You are given two hats and *four marbles.* Two of the marbles are red and two are white. You may put them into the two hats in any way you choose.	

Two Hats

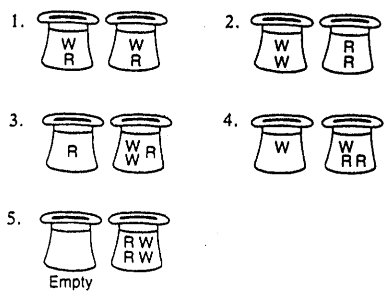
H1 H2

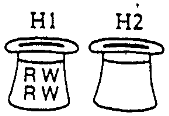
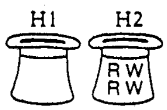


Four marbles

R R W W

TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE
	After you have placed the marbles into the hats, the hats will be taken to your friend, who will reach into one of the hats to pull out a marble. If the marble is white, you and your friend will share the prize. Otherwise, you win nothing. This includes winning nothing if your friend chooses an empty hat. How should you arrange the marbles to optimize your chances of winning the prize?	
Students need Worksheet 6-1, Which Is Best? If possible, provide colored pens for the students to mark areas with so that their work shows on the black grid.	On your worksheet you have several grid squares to use in analyzing this problem. First list every possible way that you could arrange the marbles in the hats.	
Take suggestions from the class until five *different* possibilities are found.	Mark your worksheets as I mark the overhead.	
	Now find the probabilities for each case.	



OBSERVATIONS	POSSIBLE RESPONSES
	If a student has trouble getting started, ask what all the possible arrangements are.
	Mark the hats with each arrangement.
	How can we represent the probability of picking a hat?
	In hat 1, how can I represent the probability of picking a red marble? How can I represent the probability of picking a white marble?
	What are all the outcomes (denominator)? The denominator equals the total number of squares.
	How many squares represent red (numerator)?
	Suggest that students add another grid.
A student may claim there are six arrangements by saying	Suggest that more advanced students find the best arrangement for three red and three white marbles. What happens if we add a third hat?
this  is different	With two hats  gives $\frac{7}{10}$.
from this 	With three hats  gives $\frac{9}{12}$ or $\frac{3}{4}$.

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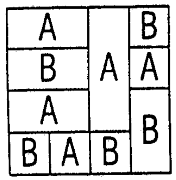
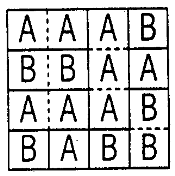
TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE		
Discuss the results of Worksheet 6-1.	What is the probability of each arrangement?	**Hat 1**	**Hat 2**	
		WW		$P(R) = \frac{1}{4}$
		RR		$P(W) = \frac{1}{4}$
		WW	RR	$P(R) = \frac{1}{2}$
				$P(W) = \frac{1}{2}$
		WR	WR	$P(R) = \frac{1}{2}$
				$P(W) = \frac{1}{2}$
		W	WRR	$P(R) = \frac{1}{3}$
				$P(W) = \frac{2}{3}$
		R	WWR	$P(R) = \frac{2}{3}$
				$P(W) = \frac{1}{3}$
	Which is the best arrangement?	H1: W and H2: WRR		

TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE
Pass out Worksheet 6-2, Darts, Anyone? Illustrate the method of attacking these problems, which is to partition the area into equal squares. Put problem 1 on the board or the overhead. 	What lines should I add to have equal areas?	Various answers.
Mark student responses on the grid. 	One good way to approach these problems is to first make equal partitions horizontally and then vertically (or vice versa).	
	How many equal areas (squares) do we have?	16.
	What is the probability of A?	$P(A) = \frac{9}{16}$
	What is the probability of B?	$P(B) = \frac{7}{16}$
	Is this fair?	No.
	How can we make it fair?	**Change one A square to a B square or let the A player get 7 points and the B player get 9 points.**
Discuss the problems on Worksheet 6-2, giving correct solutions or having students give correct solutions. Let some students show their paths and the analysis for the maze in part II, or have students exchange papers and try to figure out each others' probabilities.	In problem 2 on Worksheet 6-2, when we make the horizontal lines needed and then the vertical lines needed, we see that the basic area into which we partition the board is smaller than *any* of the original areas.	**If students have trouble weighting games in problems 3 and 4 on Worksheet 6-2, remind them that they want P(X points) for one player to equal P(X points) for the other(s). So in problem 4 we want to put numbers in □ and △ so that** $\frac{7}{16} \times \square = \frac{9}{16} \times \triangle$.

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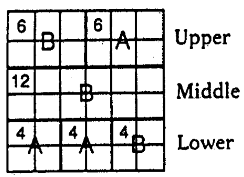
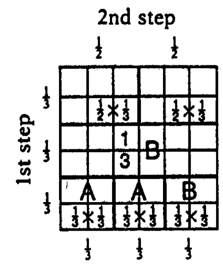
TEACHER ACTION	TEACHER TALK	EXPECTED RESPONSE
Extra challenge.	Suppose you have RRR, WWW, and three hats to place them in. How would you arrange them to optimize the probability of drawing a red?	 $P(R) = \frac{3}{4}$

You may want to relate the analysis to the multiplication of fractions. If your students are ready to multiply fractions, you can make the connection here between the area model and multiplication. As an example, the Maze (Materials 6-1) path analysis looked like this.



We can assign probabilities to areas by conducting a two step analysis and multiplying. Mark each area with the letter and number of squares as you partition.



$$P(A) = \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right)$$

$$= \frac{1}{6} + \frac{1}{9} + \frac{1}{9}$$

$$= \frac{3}{18} + \frac{2}{18} + \frac{2}{18} = \frac{7}{18}$$

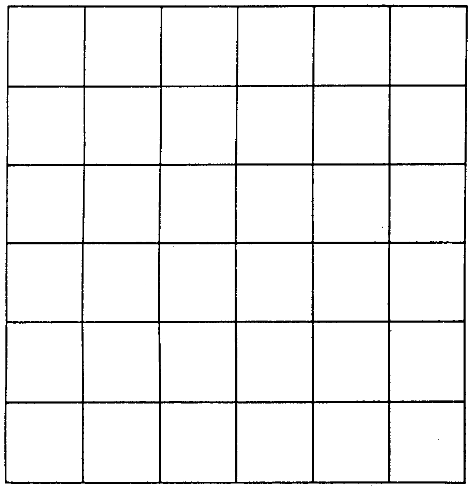
$$P(B) = \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right)$$

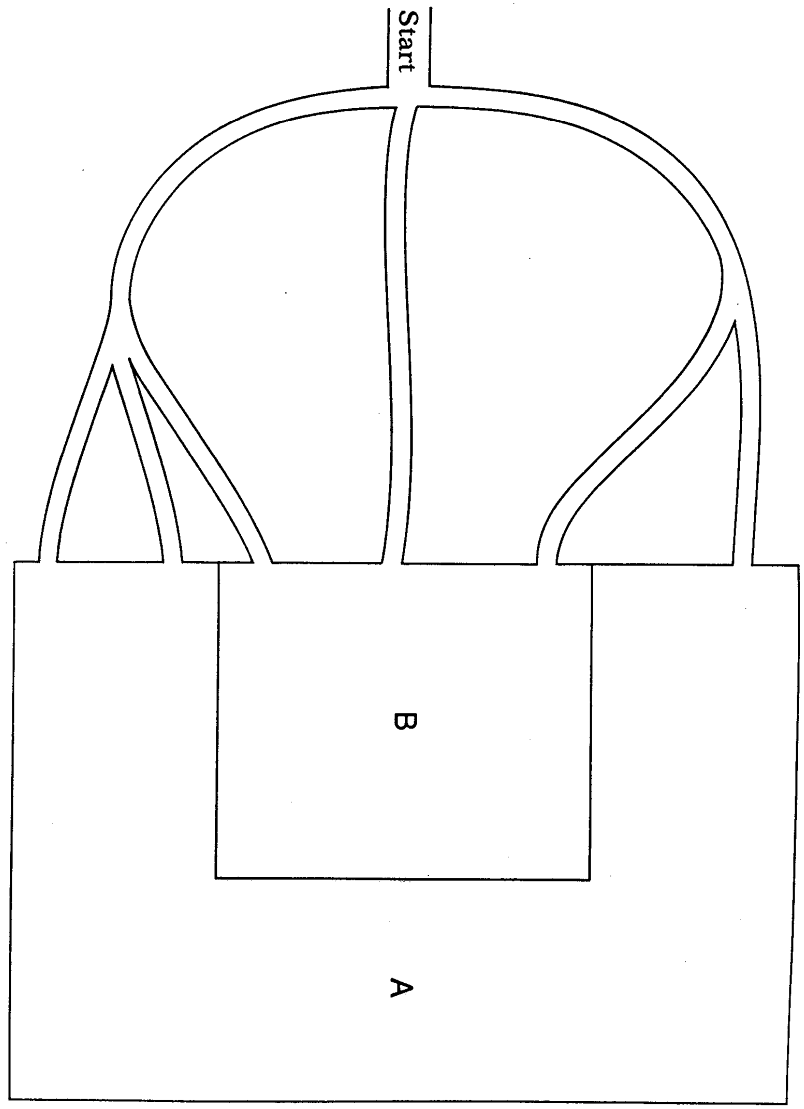
$$= \frac{1}{6} + \frac{1}{3} + \frac{1}{9}$$

$$= \frac{3}{18} + \frac{6}{18} + \frac{2}{18} = \frac{11}{18}$$

These are the same answers that we got by counting the squares in the area models.

Materials 6-1 The Maze





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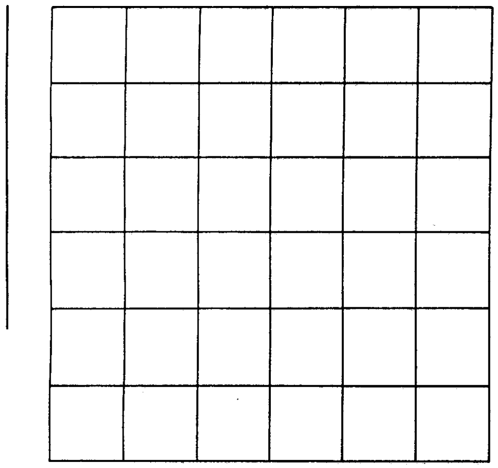
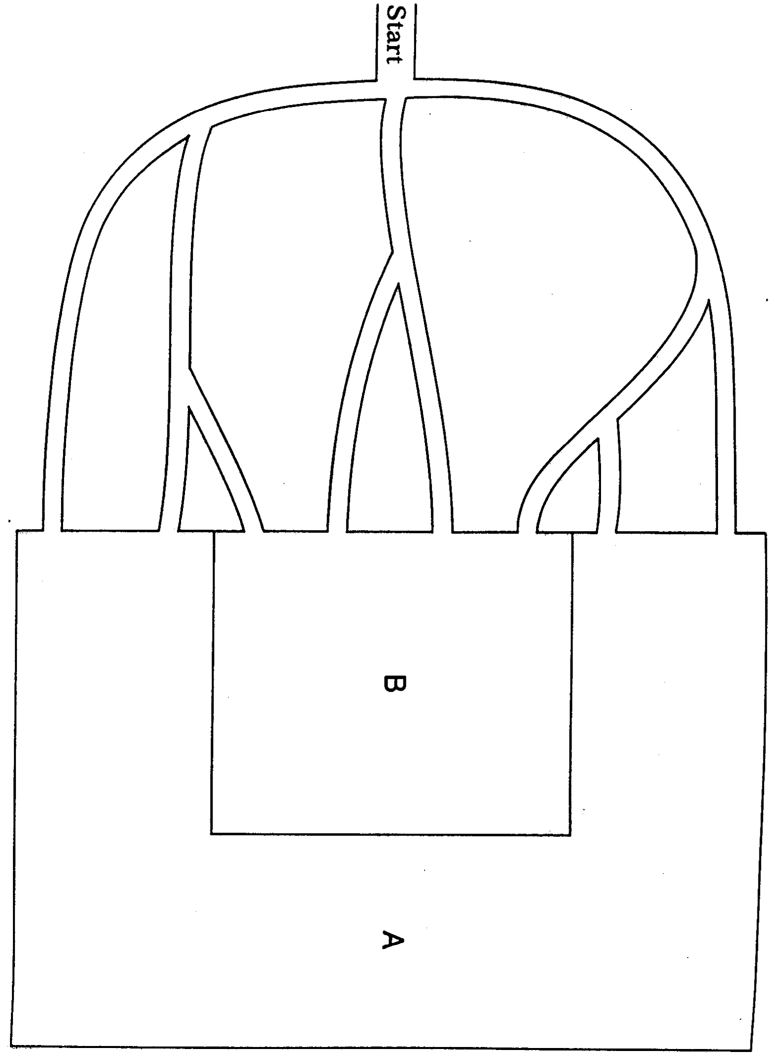
Materials 6-2

Another Maze

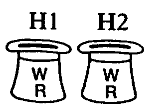




NAME ____________________

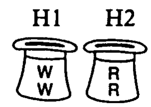
Which Is Best?

You are given two hats, two white marbles, and two red marbles. Which arrangement of the two white and two red marbles in the hats gives the best chance of drawing a white marble?



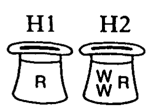
P(W) = ______

P(R) = ______



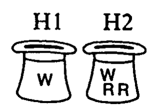
P(W) = ______

P(R) = ______



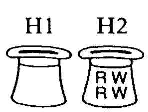
P(W) = ______

P(R) = ______



P(W) = ______

P(R) = ______



P(W) = ______

P(R) = ______

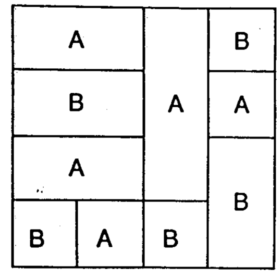
Worksheet 6-1

NAME ____________

Darts, Anyone?

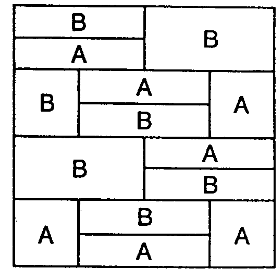
1. Pat and Erin are playing a game with the board shown at the right. A dart is thrown at random at the board. Pat scores a point if the dart lands in an area marked A. Erin scores a point if the dart lands in an area marked B. Is this a fair game?

P(A) = ________ P(B) = ________



2. Find probabilities for this board. Would this board make a fair dart game?

P(A) = ________ P(B) = ________

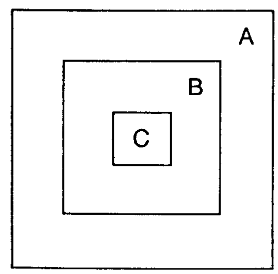


3. If a dart is thrown at random at this dart board, what is the probability that it will land in area A? What is the probability it will land in area B? What is the probability it will land in area C?

P(A) = ________ P(B) = ________ P(C) = ________

Scoring: If a dart landing in A scores one point, how many points should a dart landing in C score to make the two areas yield the same number of points over the long run? What should a dart in area B score?

Points for C ________ Points for B ________

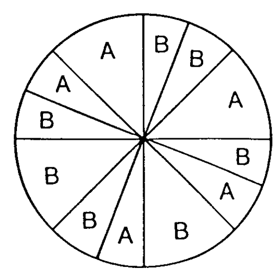


4. What is the probability that a dart thrown at random at this board will land in area A? What is the probability it will land in area B?

P(A) = ________ P(B) = ________

How would you assign points so that the game would be fair?

Points for A ________ Points for B ________

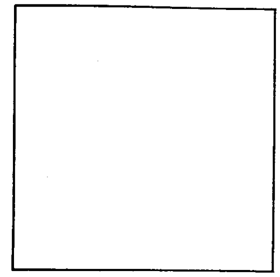


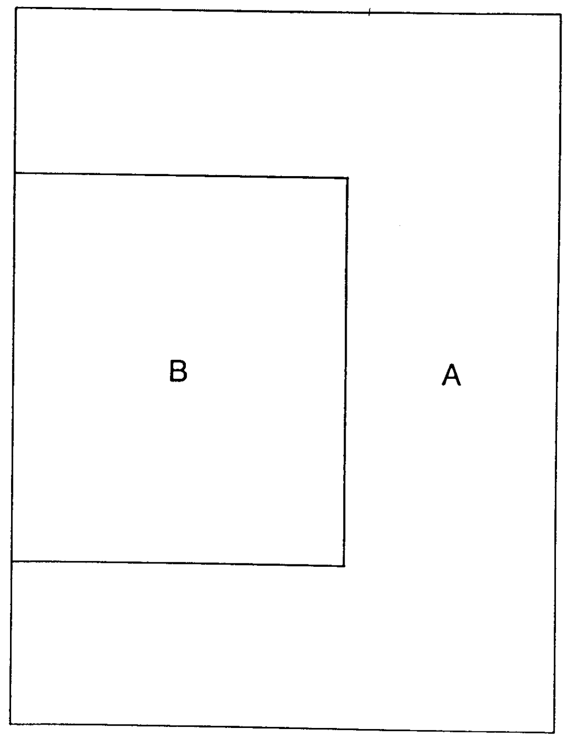
Worksheet 6-2

NAME ____________________

Darts, Anyone?

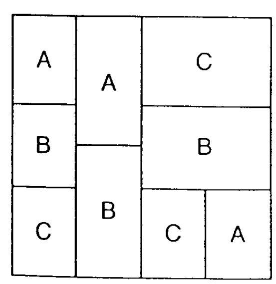
5. In the space below design a path for the story about Mr. Green and the students. Use the grid to analyze your path. Find the room in which the students should place the cookbook.





6. Jane used this grid to analyze a probability problem. What probabilities should be assigned to A, B, and C?

P(A) = ________ P(B) = ________ P(C) = ________



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