Teaching Computational Estimation: Concepts and Strategies

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The very existence of this yearbook as well as the information presented in its various articles documents the support among educators for teaching estimation in elementary and secondary school classrooms. Bridging the gap between knowing that something should be taught and actually teaching it, however, requires concrete answers to questions like these: What specific skills should be taught? How should these skills be sequenced and merged into the existing mathematics curriculum? How should the topic be presented and practiced?

This article will highlight two important features of a comprehensive estimation curriculum: the development of number concepts and the development of estimation strategies. (Many of the ideas and activities illustrated were developed through a National Science Foundation grant entitled "Developing Computational Estimation in the Middle Grades." A complete copy of these materials is available through ERIC [numbers ED 242 526, ED 242 527, and ED 242 528].

THE CONTENT OF AN ESTIMATION CURRICULUM

Estimation, much like problem solving, calls on a variety of skills and is developed and improved over a long period of time; moreover, it involves an attitude as well as a set of skills. Like problem solving, estimation is not a topic that can be isolated within a single unit of instruction. It permeates many areas of our existing curriculum, and to be effectively developed, it must be nurtured and encouraged throughout the study of mathematics. When it is taught as an isolated topic, as we have seen in recent years, the effort may even be counterproductive - leaving students with a general dislike and distrust for the very process. To be truly effective, a careful integration of estimation must occur. A comprehensive estimation curriculum must address several areas:

1. Development of an awareness for, and an appreciation of, estimation.
2. Development of number sense.
3. Development of number concepts.

The first two of these areas are discussed in detail in Trafton's article in this yearbook. The latter two will be discussed and illustrated here.

DEVELOPING NUMBER CONCEPTS

As illustrated in other articles of this yearbook, the results from the National Assessment of Educational Progress (NAEP) estimation items were very disappointing and point out the weak estimation skills of our students as well as a basic conceptual misunderstanding of fractions, decimals, and percents. Unless this conceptual error is corrected, students will continue to struggle with estimating.
Why did 76 percent of 13-year-olds incorrectly estimate the sum of 12/13 and 7/8? Is it because they didn't know how to estimate or because they didn't really understand what 12/13 and 7/8 represent? Very likely it is a combination of the two. How, then, can we help students better develop the notion of fractions, decimals, and percents?

Estimation offers an alternative way of developing concepts related to these numbers. For example, figure 3.1 illustrates a presentation that encourages students to compare numerator and denominator in order to get an estimate for fractions that are less than 1. Emphasis is placed...
on understanding the size of a fraction relative to 0, 1/2, and 1 before operating with it. This type of presentation can be done before computation with fractions is introduced. The work done in developing estimation with fractions will complement and support later work with computation. Once this basic conceptual development is established, exercises such as estimating \(\frac{12}{13} + \frac{7}{8}\) become simple (see fig. 3.2). A similar format can be used when discussing decimals that are less than 1, as illustrated in figure 3.3. Only after carefully led discussion do students begin to understand that the number of digits contained in a decimal that is less than 1 has little to do with its size. (My students seem to be conditioned to think that "more digits means bigger"!)

Another trouble spot for most junior high school students is percent. This topic is discussed in article 14.

The suggestions made here are not new. The emphasis is on instruction grounded on what the numbers in a problem mean. To me, it is much more important that students understand the meaning of numbers such as \(\frac{7}{8}\), .48, and 74% than be able to recall the appropriate algorithm to compute with them. In the long run, the algorithms will be forgotten. Numbers like these, however, will be encountered often, and, generally, a good understanding of them together with an ability to estimate will satisfy most needs. It is my observation that time spent developing these basic concepts through a mental computation and estimation approach greatly enhances, and gives meaning to, later work with exact computation.

**DEVELOPING ESTIMATION STRATEGIES**

Although there are many components to a complete estimation program, one of the most important is the teaching of estimation strategies. The mathematics curriculum of the 1970s and early 1980s included rounding as the core estimation strategy or, more usually, the only strategy. Although this is an important and useful strategy, it is not the most efficient one for many problems. When students and adults who have been identified as good estimators in a recent study were asked to estimate, they used a variety of strategies. They chose these strategies to fit...
the context of the problem, including the specific numbers and operations involved. Again, this paralleled what we know about problem solving. No one problem-solving strategy is efficient for every problem. Part of the task of becoming a good problem solver (or estimator) is being able to select and use a strategy that fits the problem. Recent research has identified several broad strategies that are used by self-developed good estimators. These include the following:

1. Front-end
2. Clustering
3. Rounding
4. Compatible numbers
5. Special numbers

Each of these strategies is best suited to a certain type of problem or operation, and several overlap in their application. Each will be briefly explained and illustrated. A sample format for presenting several of the strategies in a classroom setting will also be included. This format includes a teacher-led discussion of the strategy followed by guided practice.

**Front-End Strategy**

This strategy is one that even young students can learn to use. Although it can be modified for each of the four main operations, its strongest application is for addition. The focus is on the "front end", or left-most digits, of a number. Because these digits are the most significant, they are the most important for forming an estimate. Front-end strategy is a two-step process. For example, estimate the total of the following:

```
1. FRONT-END... Total the front-end (dollar) amounts.
   1 + 4 + 1 + 2 = $8

2. ADJUST... Group the cent amounts to form dollars.
   26 and 79 make $1
   99 cents make $1
   37 and 58 make $1

So, $8 + $3 = $11
```

This strategy can be introduced first by using money (as shown here); then other types of numbers can be substituted. The process works well with whole numbers as well as with fractions and decimals.

[Source: Estimation & Mental Computation-1986 Yearbook, National Council of Teachers of Mathematics, Permission Granted]
The adjustment step is a powerful tool for refining an initial estimate; it can be suited to the particular mathematical skill of the user. The real strength of this strategy is in the minimal amount of mental computation involved (only single-digit addition) and the fact that all numbers involved are visible to the user. It works efficiently when applied to a two-addend problem or a six-addend problem. It can be introduced as early as third grade, where the adjustment step may be delayed or treated very casually. For example, to estimate $326 + 214 + 145$, a young student could think, "$3 + 2 + 1$ is 6, so my estimate is a bit more than 600."

As mentioned earlier, the front-end strategy can be applied to subtraction (436 - 285 ... 4 - 2 is 2, so the estimate is 200, but really less, 200-). However, since with subtraction we are operating with only two numbers at a time, rounding may be at least as efficient.

An illustration of front-end multiplication will likely remind readers that they themselves have used an extended form of this process to mentally compute an exact product.

<table>
<thead>
<tr>
<th>Front-End</th>
<th>Adjustment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 736$ ... $4 \times 700 = 2800$</td>
<td>$4 \times 36$ makes at least 100</td>
</tr>
<tr>
<td>So, $4 \times 736$ is about 2900.</td>
<td></td>
</tr>
</tbody>
</table>

The traditional division algorithm is a front-end process. Estimating quotients often produces errors that result in too many or too few digits. Instruction should focus on this trouble spot. For example, estimate the quotient of $7 \sqrt{3684}$.

**Front-End Process**

1. Place the first digit of the quotient
   \[5\]
   $7 \sqrt{3684}$

2. Determine the place value of the quotient
   \[500+

3. Adjust
   \[500+

[Source: Estimation & Mental Computation-1986 Yearbook, National Council of Teachers of Mathematics, Permission Granted]
The process focuses on the first digit of the quotient and the correct place value of the quotient. This process always produces an underestimate as students quickly see; however, the adjustment step brings greater accuracy to the estimate if it is needed. Figures 3.4 and 3.5 illustrate a classroom format for developing front-end for addition and multiplication.
To Find an Exact Answer...

Multiplying usually starts at the “back end” of the problem:

\[
\begin{array}{c}
5 \\
437 \\
\times 8 \\
6 \\
\end{array}
\]

\[\text{7 \times 8 is 56}\]

We work our way to the front:

\[
\begin{array}{c}
2.5 \\
437 \\
\times 8 \\
\end{array}
\]

\[\text{3496}\]

To Find a Good Estimate...

Multiplying starts at the front end:

\[
\begin{array}{c}
437 \\
\times 8 \\
\end{array}
\]

\[\text{3200 + Good estimate}\]

\[\text{3440 + Better estimate}\]

In estimating, use as many digits as will give you the kind of estimate you need.

Fig. 3.5

Clustering Strategy

The clustering strategy is suited for a particular type of problem that we often encounter in everyday experiences. It can be used when a group of numbers cluster around a common value. For example, estimate the total attendance from the following list:
World's Fair Attendance (1-6 July)

- Monday  72 250
- Tuesday  63 819
- Wednesday  67 490
- Thursday  73 180
- Friday  74 918
- Saturday  68 490

Since all the numbers are close to each other in value, we can use clustering to estimate the total attendance.

1. Estimate an "AVERAGE"  All about 70 000

2. Multiply the "AVERAGE" by the number of values … 6 x 70 000 is 420 000

The strategy can be used with problems involving whole numbers, fractions, or decimals. It eliminates the mental tabulation of a long list of front-end or rounded digits, creating instead a problem with fewer digits that are easily computed. Although clustering is limited to a certain type of problem, it involves a natural translation process and is one that many students (and adults) discover and use on their own.

Rounding Strategy

The rounding strategy is a powerful and efficient process for estimating the product of two multidigit factors. The strategy involves first, rounding numbers; and second, computing with the rounded numbers. A third step of adjustment can sometimes be added when both factors are rounded in the same direction, as illustrated in figure 3.6.

This adjustment of an estimated product is a natural process. I've had a lot of rich discussion with junior high school students about how a product should be adjusted if one factor is rounded up and one down. I always point out that if each number has been rounded in an opposite direction, the process of adjustment has already been accomplished internally and the resulting product is quite satisfactory without further adjustment.

Instruction on this strategy should clearly point out the situations when rounding can be used efficiently. It should also emphasize that numbers can be rounded in many ways, as illustrated in the following examples:
Adjusting Estimates

**Fig. 3.6**

- **28 \times 56**
  - **Round Up**
  - 30 \times 60
  - **1800 is an overestimate.** So, I’ll adjust down: 1800-

- **62 \times 23**
  - **Round Down**
  - 60 \times 20
  - **1200 is an underestimate.** So, I’ll adjust up: 1200 +

- **62 \times 79**
  - **Down Up**
  - 60 \times 80
  - You can’t really tell. So, I’ll just say... 4800.

- **36 \times 75**
  - **Since they are both close to the middle, I’ll round one up and one down:** 2800.
Each rounding choice produces different but reasonable estimates. The choice of rounded factors will be dictated by the user's flexibility and ability to compute mentally.

Students should remember that the purpose of the rounding strategy is to produce mentally manageable numbers. They need to learn to be flexible in their method of rounding - fitting it to the particular situation, operation, and numbers involved. This experience will strengthen the understanding that estimation is a simplifying process. Further, it will help students appreciate that estimators decide for themselves which simpler problem is most comfortable and accurate for their purposes.

**Compatible Numbers Strategy**

The compatible numbers strategy refers to a set of numbers that can be easily "fit together" (i.e., are easy to manipulate mentally). It encourages the user to take a global look at all numbers involved in a problem and to change or round each number so it can be paired usefully with another number. The choice of the particular set of compatible numbers involves a flexible rounding process. This strategy is particularly effective when estimating division problems. For example:

<table>
<thead>
<tr>
<th>Estimate:</th>
<th>These are compatible sets:</th>
<th>These are not compatible sets:</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 (\sqrt{3388})</td>
<td>7 (\sqrt{3500})</td>
<td>7 (\sqrt{3000})</td>
</tr>
<tr>
<td>8 (\sqrt{3200})</td>
<td>8 (\sqrt{4000})</td>
<td>7 (\sqrt{3300})</td>
</tr>
</tbody>
</table>

The compatible numbers strategy can also be used for addition problems with several addends. The student learns to look for pairs of numbers that "fit together" to make numbers that are easy to compute mentally. Figure 3.7 illustrates this strategy for addition. This strategy involves a certain level of sophistication, experience, and flexibility.
Special Numbers Strategy

This strategy overlaps several already discussed. Students are encouraged to be on the lookout for numbers that are near "special" values that are easy to compute mentally. Special values include powers of ten and common fractions and decimals. For example, each of these problems involves numbers near special values, and therefore they can be easily estimated.

<table>
<thead>
<tr>
<th>Problem:</th>
<th>Think:</th>
<th>Estimate:</th>
</tr>
</thead>
<tbody>
<tr>
<td>7/8 + 12/13</td>
<td>Each near 1</td>
<td>1 + 1 = 2</td>
</tr>
<tr>
<td>23/45 of 720</td>
<td>23/45 near 1/2</td>
<td>1/2 of 720 = 360</td>
</tr>
<tr>
<td>9.84% of 816</td>
<td>9.84% near 10%</td>
<td>10% of 816 = 81.6</td>
</tr>
<tr>
<td>.98 ( \sqrt{436.2} )</td>
<td>.98 near 1</td>
<td>436 \div 1 = 436</td>
</tr>
<tr>
<td>103.96 x 14.8</td>
<td>103.96 near 100 ( \frac{14.8}{15} )</td>
<td>100 \times 15 = 1500</td>
</tr>
</tbody>
</table>

The special numbers strategy is best taught along with the development of fraction, decimal, and percentage concepts, which were discussed earlier in this article. In a sense, special numbers and compatible numbers strategies illustrate best what estimation really is: the process of taking an existing problem and changing it into a new form that has these two characteristics:

1. Approximately equivalent answer
2. Easy to compute mentally

In some instances, numbers are changed only slightly (e.g., \( 7/8 + 12/13 \approx 1 + 1 \)). In others, more drastic changes are needed to accomplish characteristic 2 (e.g., 24% of 78 \( \approx 25\% \) of 80 \( = 80 \div 4 \)). Several applications of this strategy are seen in figures 3.8, 3.9, and 3.10.

PUTTING IT TOGETHER

Like problem-solving techniques, estimation strategies are developed through careful instruction, discussion, and use. For the best development of estimation skills, the following three phases should be included:

1. **Instruction.** Unless computation estimation strategies are taught, most students will neither learn nor use them. Prerequisite skills (such as the mastery of basic facts and place value) must be reflected in the instruction and development of a strategy. Greater understanding and appreciation of a strategy will result when it is related to different applied situations. Practice is important, but instruction on each of these estimation strategies will complement, direct, and promote meaningful practice.
Estimating Sums of Fractions

Claudia jogged \( \frac{1}{2} \) mile in the morning and \( \frac{3}{8} \) mile in the afternoon. 

Did she jog at least 1 mile?

\[ \frac{3}{8} \text{ is close to } \frac{1}{2}, \text{ but a little less.} \]

So, Claudia jogged less than 1 mile.

Try These:

\[ \frac{7}{8} + \frac{5}{9} \quad \frac{4}{5} + \frac{5}{9} \quad \frac{4}{9} + \frac{3}{8} \]

About \( \frac{2}{2} \)

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Estimate:

\( \frac{23}{49} \) of 720

Clean the problem up. Think...

- What common fraction is \( \frac{23}{49} \) near?

  It's about \( \frac{1}{2} \)!

  - Change it.

    \( \frac{1}{2} \) of 720

    - Compute.

    That's 360!

Let's practice. Name a common fraction near each of these:

\[ \frac{4}{9} \quad \frac{5}{14} \quad \frac{17}{18} \quad \frac{14}{45} \quad \frac{7}{29} \quad \frac{16}{30} \]
2. **Practice.** It is important to have a wide variety of practice preceded by specific instruction. Short practice sessions of five to ten minutes each week are recommended. Such regular practice will help maintain basic facts, improve mental computation skills, and provide opportunities for further development of computational estimation skills.

3. **Testing.** Periodic testing provides motivation for developing computational estimation. Each test can include about a dozen similar items. An effective format for presentation is to put the items on an overhead transparency and project each problem individually for a short period of time (ten to twenty seconds). Scoring intervals can be set up in advance for each problem, and then selected problems and the strategies used to solve them can be discussed in class.

Estimation is a rich topic and is fun to teach. It offers many avenues of discussion with students which overlap as well as support many other concepts we already teach. The ideas presented here are intended to get you started. Many materials on estimation are becoming available to help guide your instruction. For additional ideas, see the sources listed in the Bibliography.
BIBLIOGRAPHY


